

Quasiconformal dilatation of projective transformations and discrete conformal maps

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Abstract

We consider the quasiconformal dilatation of projective transformations of the real projective plane. For non-affine transformations, the contour lines of dilatation form a hyperbolic pencil of circles, and these are the only circles that are mapped to circles. We apply this result to analyze the dilatation of the circumcircle preserving piecewise projective interpolation between discretely conformally equivalent triangulations. We show that another interpolation scheme, angle bisector preserving piecewise projective interpolation, is in a sense optimal with respect to dilatation. These two interpolation schemes belong to a one-parameter family.

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1 Overview and motivation

The deviation of a function from being conformal is measured by its quasiconformal dilatation. (Standard ways to quantify the local deviation from conformality will be reviewed in Section 2.) This paper is about the quasiconformal dilatation of real projective transformations of the plane. In Section 3 we show that the contour lines of dilatation form a hyperbolic pencil of circles (Theorem 3.1). These are the only circles that are mapped to circles (Theorem 3.4). Although the statements are elementary and the proofs are straightforward, these results seem to be new. It may well be that the dilatation of a real projective map was never considered before. We were motivated by the theory of discrete conformal maps that is based on the following definition of discrete conformal equivalence of triangle meshes. (Quasiconformal dilatation also plays an important role in at least one other theory of discrete conformality, circle packing [8].)

1.1 Definition and Theorem ([3, 5]). Two triangulated surfaces are considered *discretely conformally equivalent*, if the triangulations are combinatorially equivalent and if one (and hence all) of the following equivalent conditions hold (see Figure 1 for notation).

- (i) The edge lengths ℓ_{ij} and $\tilde{\ell}_{ij}$ of corresponding edges are related by

$$\tilde{\ell}_{ij} = e^{\frac{1}{2}(u_i + u_j)} \ell_{ij}$$

for some logarithmic scale factors u_i associated to the vertices.

- (ii) For interior edges ij , the *length cross ratios* are equal, that is,

$$\frac{\ell_{im}\ell_{jk}}{\ell_{mj}\ell_{ki}} = \frac{\tilde{\ell}_{im}\tilde{\ell}_{jk}}{\tilde{\ell}_{mj}\tilde{\ell}_{ki}}.$$

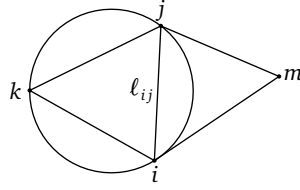


Figure 1: Notation for the definition of discrete conformal equivalence.

- (iii) The *circumcircle preserving projective maps* that map a triangle of one triangulation to the corresponding triangle of the other triangulation, and the respective circumcircles onto each other, fit together continuously across edges.

The original definition (i) is due to Feng Luo [5]. For the equivalent characterizations (ii) and (iii), see [3]. Characterization (iii) means that discretely conformally equivalent triangle meshes allow not only the usual piecewise linear (PL) interpolation (which works for any two combinatorially equivalent triangulations) but also *circumcircle preserving piecewise projective (CPP) interpolation* (which is not continuous across edges unless two triangulations are discretely conformally equivalent). This motivated the following definition.

1.2 Definition ([3]). A simplicial continuous map between triangulated surfaces is a *discrete conformal map* if the restriction to any triangle is a circumcircle preserving projective map onto the image triangle.

Figure 2 shows visualizations of PL and CPP interpolations. The CPP interpolations clearly look better. (This is an important point for applications in computer graphics [7].) What is the reason? This question led us to investigate the quasiconformal dilatation of projective transformations. We wanted to check the hypothesis that the CPP interpolations had lower dilatation. Behind this hypothesis was the non-mathematical hypothesis that lower dilatation was the reason why CPP interpolation looks better; however, see Remark 6.4 in Section 6.

As we show in Section 4, the dilatation of CPP interpolation is indeed pointwise less than or equal to the dilatation of PL interpolation, but the maximal dilatations per triangle are equal (Corollary 4.2).

In Section 5, we show that another interpolation scheme, mapping angle bisectors to angle bisectors, is in a sense optimal with respect to dilatation (Theorem 5.1). Like CPP interpolation, angle bisector preserving piecewise projective (APP) interpolation is continuous across edges if and only if the triangulations are discretely conformally equivalent (Theorem 5.3).

The interpolation schemes PL, CPP, and APP are in fact members of a continuous one-parameter family (see Section 6). We do not know any interesting geometric characterization for any other member of this family.

2 Dilatation and eccentricity

In this section, we review the definitions of dilatation and eccentricity of real differentiable maps. The definitions are standard (the provided references are only meant as examples), but our perspective is slightly unusual because (in Section 3) we are interested in maps that become singular and orientation reversing. (Real projective transformations are generally not quasiconformal functions on \mathbb{C} .)

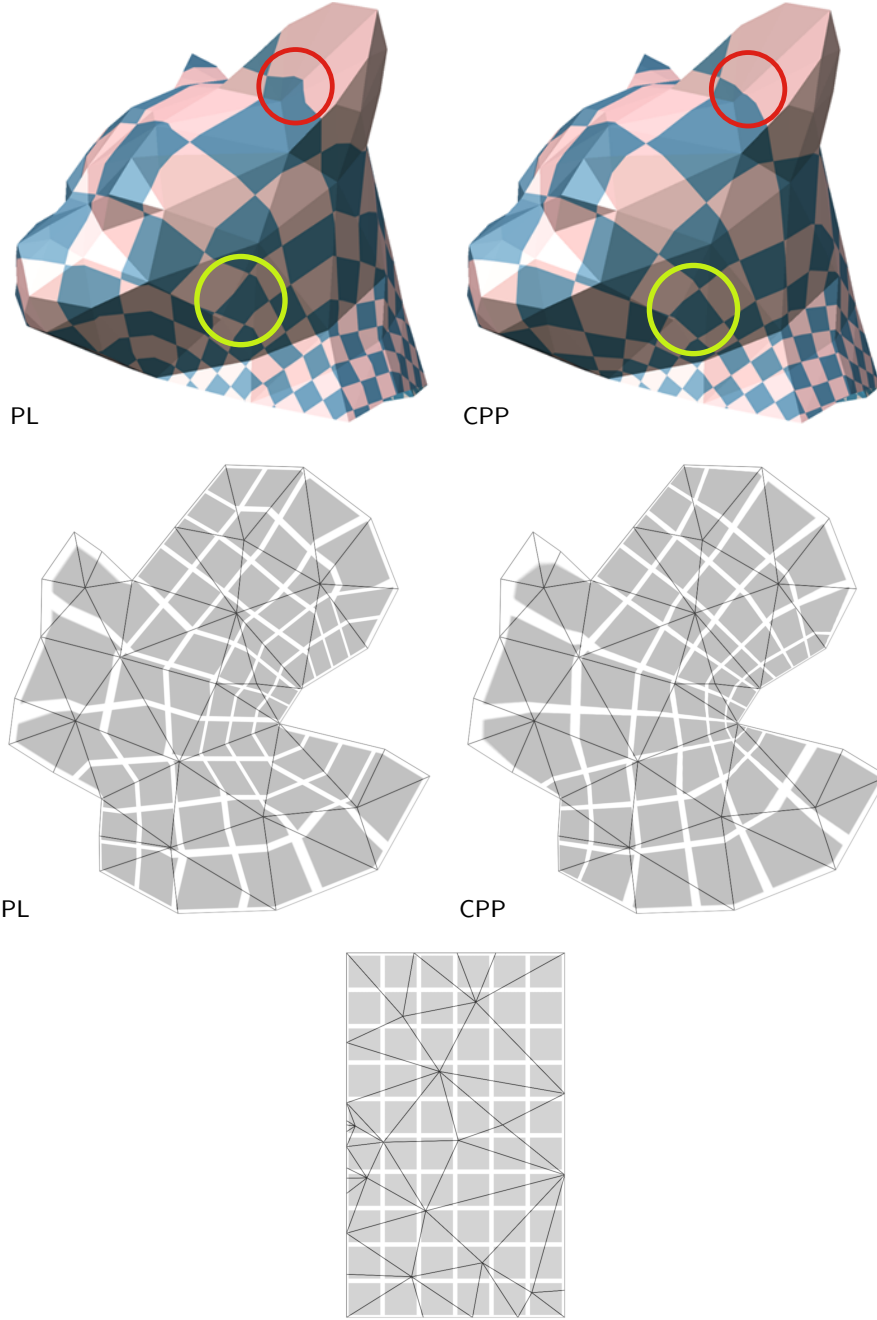


Figure 2: Piecewise linear (PL) vs circumcircle preserving piecewise projective (CPP) interpolation. *Top row* [7]: A triangulated surface in \mathbb{R}^3 resembling a cat head is mapped to a conformally equivalent planar triangulation (not shown) by PL and CPP interpolation. The maps are visualized by pulling back a checkerboard pattern in the plane to the cat head. Some notable differences are highlighted. *Middle row* [3]: A triangulated planar region is mapped to a conformally equivalent triangulation of a rectangle (*bottom*) by PL and CPP interpolation.

In the following, let $U \subseteq \mathbb{C}$ be open and let $f : U \rightarrow \mathbb{C}$ be a real differentiable map. (We identify \mathbb{C} and \mathbb{R}^2 as euclidean vector spaces.) The function f maps a small circle of radius ε around $z \in U$ approximately to an ellipse with major semi-axis $\lambda_1 \varepsilon$ and minor semi-axis $\lambda_2 \varepsilon$, where

$$\lambda_1 \geq \lambda_2 \geq 0$$

are the singular values of the real derivative df_z of f at z .

The quotient of singular values is usually called the *dilatation* [1, 2], or, to be more specific, the *quasiconformal dilatation* [6]. However, since we are interested in maps that may become singular and orientation reversing, we define the *[signed] dilatation* $D_f(z) \in \mathbb{R} \cup \{\infty\}$ of f at the point $z \in U$ by

$$D_f(z) = \pm \frac{\lambda_1}{\lambda_2},$$

where the sign is chosen according to whether f is orientation preserving (+) or reversing (−) at z . If the derivative df is singular at z , then $\lambda_2 = 0$ and we define $D_f(z) = \infty$. But we will assume that df vanishes nowhere, so the dilatation is well defined everywhere. Note the following properties of the dilatation:

- $|D_f| \geq 1$,
- $D_f = \pm 1$ where f is conformal or anticonformal, respectively,
- $D_f(z) = D_{f^{-1}}(f(z))$.

The *Beltrami coefficient*

$$\mu_f(z) = f_{\bar{z}}/f_z \tag{1}$$

is defined in terms of the Wirtinger derivatives

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

The modulus of the Beltrami coefficient is called the *eccentricity* of f [1]. The eccentricity $|\mu_f|$ is related to the dilatation D_f by

$$|\mu_f| = \frac{D_f - 1}{D_f + 1}.$$

Note that $|\mu_f| \in [0, \infty]$, and

$$|\mu_f| = \begin{cases} 0 & \text{where } f_{\bar{z}} = 0, \text{ that is, where } f \text{ is conformal,} \\ \infty & \text{where } f_z = 0, \text{ that is, where } f \text{ is anticonformal,} \end{cases}$$

$$|\mu_f| \leq 1 \quad \text{where} \quad \det df \geq 0,$$

and

$$|\mu_f(z)| = |\mu_{f^{-1}}(f(z))|. \tag{2}$$

2.1 Remark (on terminology). (i) A function is called *quasiconformal* if it is orientation preserving and its dilatation is bounded. In phrases like “quasiconformal dilatation”, the adjective “quasiconformal” is used somewhat sloppily to mean “belonging to the theory of quasiconformal functions”. (ii) The above definition of “eccentricity” has fallen into disuse. The Beltrami coefficient μ_f is also called *complex dilatation* [2], although *complex eccentricity* would make more sense.

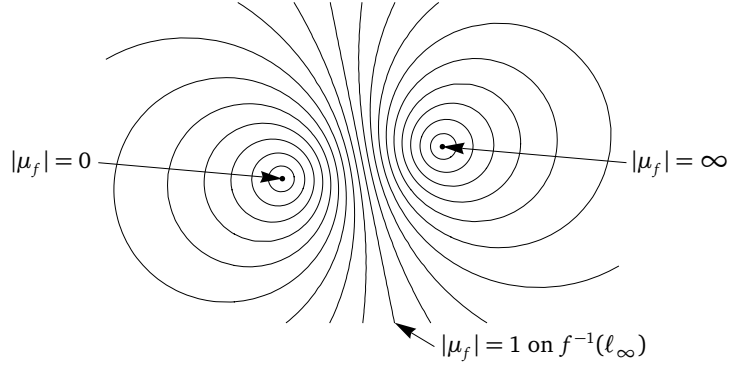


Figure 3: The contour lines of $|\mu_f|$ are a hyperbolic pencil of circles (Theorem 3.1).

3 Dilatation of a projective map, and circles mapped to circles

We are interested in the dilatation of a projective map

$$f : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2, \quad [x] \mapsto [Ax], \quad A \in \text{GL}_3(\mathbb{R}),$$

where we identify the complex plane \mathbb{C} with the real projective plane \mathbb{RP}^2 without the line $\ell_\infty = \{[x] \mid x_3 = 0\}$ via the map

$$z = x + iy \longleftrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

This provides the conformal structure on $\mathbb{RP}^2 \setminus \ell_\infty$. We consider the line ℓ_∞ as the line at infinity. The map f is [real] affine if it maps ℓ_∞ to ℓ_∞ . The map f is a similarity transformation if it is real affine and complex affine on \mathbb{C} , that is, $z \mapsto az + b$ with $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$.

The eccentricity of affine maps is constant, and identically 0 for similarity transformations. The following theorem treats the interesting case (see Figure 3).

3.1 Theorem. *If the projective transformation $f : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ is not affine then:*

- (i) *The contour lines of $|\mu_f|$ are a hyperbolic pencil of circles.*
- (ii) *The function $|\mu_f|$ is an affine parameter for this 1-parameter family of circles that assigns the values 0 and ∞ to the circles that degenerate to points and the value 1 to the preimage of the line at infinity under f .*
- (iii) *The function f maps this hyperbolic pencil of circles to another hyperbolic pencil of circles*

The following proof relies on direct calculations. It would be nice to have a more conceptual argument.

Proof. In terms of the affine parameter z on $\mathbb{RP}^2 \setminus \ell_\infty$, the projective map f may be written as

$$z \mapsto \frac{az + b\bar{z} + c}{pz + \bar{p}\bar{z} + q}, \quad a, b, c, p \in \mathbb{C}, \quad q \in \mathbb{R}, \quad (3)$$

where $p \neq 0$ because f is not affine. In terms of the coefficients a, b, c, p, q , the matrix representing f is

$$A = \begin{pmatrix} a_1 + b_1 & -a_2 + b_2 & c_1 \\ a_2 + b_2 & a_1 - b_1 & c_2 \\ 2p_1 & -2p_2 & q \end{pmatrix},$$

where indices 1 and 2 to denote real and imaginary parts, respectively.

From the definition (1) of μ , using the representation (3) for f , one obtains immediately

$$\mu_f(z) = \frac{\alpha z + \beta}{-\alpha \bar{z} + \gamma}, \quad \text{where} \quad \begin{cases} \alpha = bp - a\bar{p} \\ \beta = bq - c\bar{p} \\ \gamma = aq - cp \end{cases},$$

so

$$|\mu_f(z)| = \left| \frac{\alpha z + \beta}{-\alpha \bar{z} + \gamma} \right|.$$

As a tedious but straightforward calculation shows,

$$\det \begin{pmatrix} \alpha & \beta \\ -\bar{\alpha} & \bar{\gamma} \end{pmatrix} = \bar{p} \det A \neq 0,$$

so $|\mu_f(z)| = |M(z)|$, where M is a Möbius transformation. Parts (i) and (ii) of the theorem follow easily.

To see part (iii), note that the inverse map f^{-1} is again a projective map which is not affine. Therefore the contour lines of $|\mu_{f^{-1}}|$ are also the circles of a hyperbolic pencil of circles. Because f and f^{-1} have the same dilatation at corresponding points (equation (2)), f maps circles with constant $|\mu_f|$ to circles with constant $|\mu_{f^{-1}}|$. This proves (iii). \square

3.2 Corollary. *If f is orientation preserving on a triangle T then the function $|\mu_f|$ attains the maximum*

$$\max_{z \in T} |\mu_f(z)|$$

at a vertex.

Indeed, in the open half-plane where f is orientation preserving, the sublevel sets of $|\mu_f|$ are strictly convex (being disks), and every point is on the boundary of its sublevel set. Hence, the maximum cannot be attained in the interior of the triangle or in the relative interior of a side.

3.3 Remark. Here and throughout this article, “triangle” shall mean “closed triangular region”.

Theorem 3.1 suggests the following question: Which circles are mapped to circles by a projective map f ? Obviously, a similarity transformation maps all circle to circles, and an affine transformation that is not a similarity maps all circle to ellipses that are not circles. The following theorem treats the remaining case.

3.4 Theorem. *If the projective transformation $f : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ is not affine then it maps precisely the circles of a hyperbolic pencil to circles.*

Proof. This proof relies on the classical characterization of circles in terms of their complex intersection points with the line at infinity. Consider the real projective plane \mathbb{RP}^2 as the set of points with real homogeneous coordinates in the complex projective plane \mathbb{CP}^2 . In \mathbb{CP}^2 , every conic section intersects the line at infinity twice (counting multiplicity). Circles are the conics intersecting the line at infinity in the *imaginary circle points*

$$K = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}.$$

The circles that f maps to circles are the conics containing $K, \bar{K}, f^{-1}(K)$, and $f^{-1}(\bar{K})$. Because f is real and not affine, these four points are in general position: No three of them are contained in a line, which would have to be the line at infinity or its preimage under f . But these lines intersect in a real point. Hence the conics containing the four points form a pencil of conics. Since the four points are two pairs of complex conjugates, the pencil is a hyperbolic pencil of real conics. \square

4 The circumcircle preserving projective map

The following theorem characterizes the maximal value of $|\mu_f|$ for a circumcircle preserving projective map on a triangle.

4.1 Theorem. *If the projective map f maps a triangle T with vertices A, B, C onto a triangle \tilde{T} with vertices $\tilde{A}, \tilde{B}, \tilde{C}$, and the circumcircle of T to the circumcircle of \tilde{T} , then*

$$|\mu_f(A)| = |\mu_f(B)| = |\mu_f(C)| = |\mu_h|, \quad (4)$$

where h is the real affine map from T onto \tilde{T} (whose eccentricity $|\mu_h|$ is constant).

Together with the results of the previous section, Theorem 4.1 implies the following corollary.

4.2 Corollary. *If a projective map f satisfying the assumptions of Theorem 4.1 is orientation preserving on T , then*

$$|\mu_f(P)| \leq |\mu_h(P)|,$$

for all points $P \in T$, and

$$\max_{P \in T} |\mu_f(P)| = |\mu_h|.$$

4.3 Remark. The projective map f of Theorem 4.1 is either orientation preserving or orientation reversing on all of T . Since f maps circumcircle to circumcircle, neither ℓ_∞ nor $f^{-1}(\ell_\infty)$ intersects T .

Proof of Theorem 4.1. Since the circumcircle is mapped to the circumcircle, it is a contour line of $|\mu_f|$ (Theorems 3.1 and 3.4). This proves the first two equalities of equation (4). It remains to show the last equality involving the eccentricity of the affine map. We will avoid any calculation of derivatives.

Given three lines ℓ_1, ℓ_2, ℓ_3 intersecting in one point P and three lines $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3$ intersecting in one point \tilde{P} , there exists an affine map F mapping ℓ_j to $\tilde{\ell}_j$ ($j = 1, 2, 3$), and this map is unique up to post-composition with a homothety centered at \tilde{P} (or, which amounts to the same, pre-composition with a homothety centered at P). The value of μ_F is therefore uniquely determined by the lines.

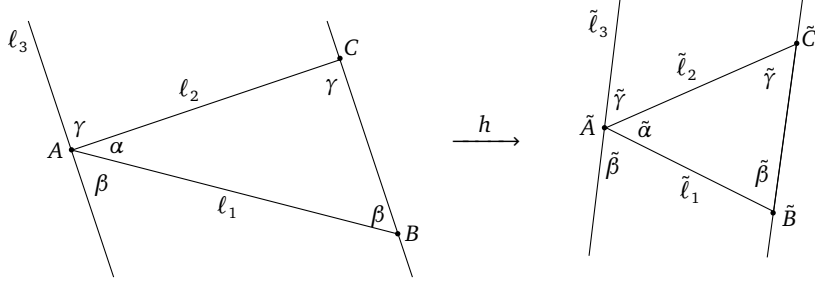


Figure 4: The affine map h from T onto \tilde{T} .

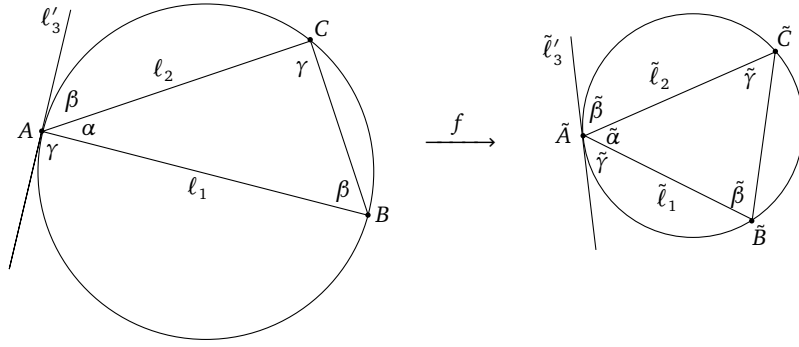


Figure 5: The circumcircle preserving map f from T onto \tilde{T} .

The affine map h maps the point A to the point \tilde{A} and the lines $\ell_1 = AB$, $\ell_2 = AC$, and the line ℓ_3 parallel to CB through A to the lines $\tilde{\ell}_1 = \tilde{A}\tilde{B}$, $\tilde{\ell}_2 = \tilde{A}\tilde{C}$, and the line $\tilde{\ell}_3$ parallel to $\tilde{C}\tilde{B}$ through \tilde{A} (see Figure 4).

The circumcircle preserving projective map f also maps ℓ_1 to $\tilde{\ell}_1$ and ℓ_2 to $\tilde{\ell}_2$, and it maps the tangent ℓ'_3 to the circumcircle at A to the tangent $\tilde{\ell}'_3$ to the circumcircle at \tilde{A} (see Figure 5). The affine approximation of f at A (that is, in affine coordinates, the first order Taylor polynomial) also maps these lines to the same lines. Considering the angles at A and \tilde{A} in Figures 4 and 5, one finds that the affine approximation of f at A is equal to h up to composition with similarity transformations. Thus, $|\mu_f(A)| = |\mu_h|$. \square

5 The angle bisector preserving projective map

Which projective transformation between two given triangles minimizes the maximal dilatation? Since the maximal dilatation is attained at a vertex (Corollary 3.2), it is enough to minimize the maximal dilatation at the vertices. As it turns out, there is a unique projective transformation that simultaneously minimizes the dilatation at all three vertices:

5.1 Theorem. *Of all projective maps that map a triangle $T \subset \mathbb{C}$ with vertices A, B, C onto a triangle $\tilde{T} \subset \mathbb{C}$ with vertices $\tilde{A}, \tilde{B}, \tilde{C}$, which are labeled in the same orientation so that the maps are orientation preserving, the one that maps the angle bisectors to the angle bisectors simultaneously minimizes $|\mu(A)|$, $|\mu(B)|$, and $|\mu(C)|$. That is, the*

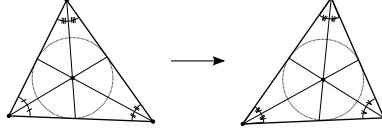


Figure 6: The angle bisector preserving projective transformation maps incircle center to incircle center, but it does in general not map incircle to incircle.

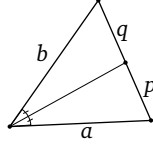


Figure 7: Angle bisector theorem: $\frac{a}{b} = \frac{p}{q}$.

angle bisector preserving projective transformation f satisfies

$$|\mu_f(A)| \leq |\mu_g(A)|, \quad |\mu_f(B)| \leq |\mu_g(B)|, \quad |\mu_f(C)| \leq |\mu_g(C)|$$

for all projective transformations g with $g(A) = \tilde{A}$, $g(B) = \tilde{B}$, $g(C) = \tilde{C}$, $g(T) = \tilde{T}$.

This theorem follows immediately from the following Lemma, whose proof we leave to the reader. (All arguments we know rely ultimately on one or another more or less direct calculation.)

5.2 Lemma. *Of all linear maps in $\text{SL}_2(\mathbb{R})$ that map two one-dimensional subspaces $\mathbb{R}v, \mathbb{R}w$ onto two one-dimensional subspaces $\mathbb{R}\tilde{v}, \mathbb{R}\tilde{w}$, the map $f \in \text{SL}_2(\mathbb{R})$ has the least dilatation if and only if it maps one, and hence both, of the angle bisectors $\mathbb{R}\left(\frac{v}{|v|} \pm \frac{w}{|w|}\right)$ to the corresponding angle bisector $\mathbb{R}\left(\frac{\tilde{v}}{|\tilde{v}|} \pm \frac{\tilde{w}}{|\tilde{w}|}\right)$. (This determines f uniquely up to sign.)*

The characterization of discrete conformal equivalence in terms of projective maps (Definition and Theorem 1.1 (iii)) remains true if “circumcircle preserving” is replaced by “angle bisector preserving”:

5.3 Theorem. *The angle bisector preserving projective maps between corresponding triangles of combinatorially equivalent triangulations fit together continuously across edges if and only if the triangulations are discretely conformally equivalent.*

Proof. This follows immediately from the elementary angle bisector theorem (Figure 7) and the characterization of discrete conformal equivalence in terms of length cross ratios (Definition and Theorem 1.1 (ii)). \square

6 A one-parameter family of piecewise projective interpolations

The angle bisector preserving projective transformation maps incircle center to incircle center. The incircle center in a triangle ABC has barycentric coordinates $[a, b, c]$, where a, b, c denote the lengths of opposite sides.

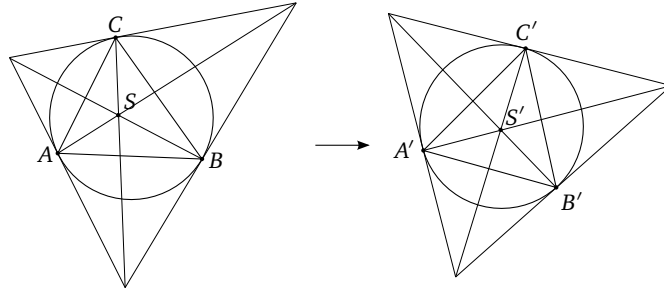


Figure 8: The circumcircle preserving projective transformation mapping ABC to $A'B'C'$ maps the point S to the point S' . The points S and S' are called the *symmedian*, *Lemoine*, or *Grebe point* of the triangles ABC and $A'B'C'$, respectively. (They are the *Gergonne points* of the triangles formed by the tangents.)

The circumcircle preserving projective transformation maps symmedian point to symmedian point (see Figure 8). The symmedian point has barycentric coordinates $[a^2, b^2, c^2]$.

In the affine case, the barycenter is mapped to the barycenter. Its barycentric coordinates are $[1, 1, 1]$.

6.1 Definition. For $t \in \mathbb{R}$, let the *exponent- t -center* in a triangle ABC be the point with barycentric coordinates $[a^t, b^t, c^t]$, where a, b, c are the lengths of opposite sides.

6.2 Remark. For parameter values different from $t = 0$ (barycenter), $t = 1$ (incircle center), and $t = 2$ (symmedian point), the exponent- t -centers of a triangle seem to be fairly esoteric triangle centers. For example, the values $t = 3, 4$ correspond to triangle centers $X(31)$ and $X(32)$ in Kimberling's list [4], and the values $t = -1, -2$ correspond to $X(75)$ and $X(76)$. If there are any other exponent- t -centers in Kimberling's list, they have indices greater than 300.

The proof of Theorem 5.3 obviously generalizes to “exponent- t -center preserving projective maps”:

6.3 Theorem. For any $t \in \mathbb{R}$, the projective maps between corresponding triangles of combinatorially equivalent triangulations that map exponent- t -centers to exponent- t -centers fit together continuously across edges if and for $t \neq 0$ only if the triangulations are discretely conformally equivalent.

This leads to a one-parameter family of exponent- t -center preserving piecewise projective interpolation schemes for discretely conformally equivalent triangulations. The parameter value $t = 0$ corresponds to piecewise linear interpolation, $t = 1$ to angle bisector preserving, and $t = 2$ to circumcircle preserving piecewise projective interpolation. Figure 9 visualizes these interpolation schemes for some values of t using the same example as Figure 2, middle.

6.4 Remark. In our eyes, circumcircle preserving piecewise projective interpolation ($t = 2$, CPP) looks slightly better than angle bisector preserving interpolation ($t = 1$, APP). We have found this in other examples as well. Since APP has in general lower maximal dilatation per triangle, this indicates that low dilatation is not what makes CPP interpolation look better. We do not know which mathematical property of CPP interpolation accounts for the nicer appearance.

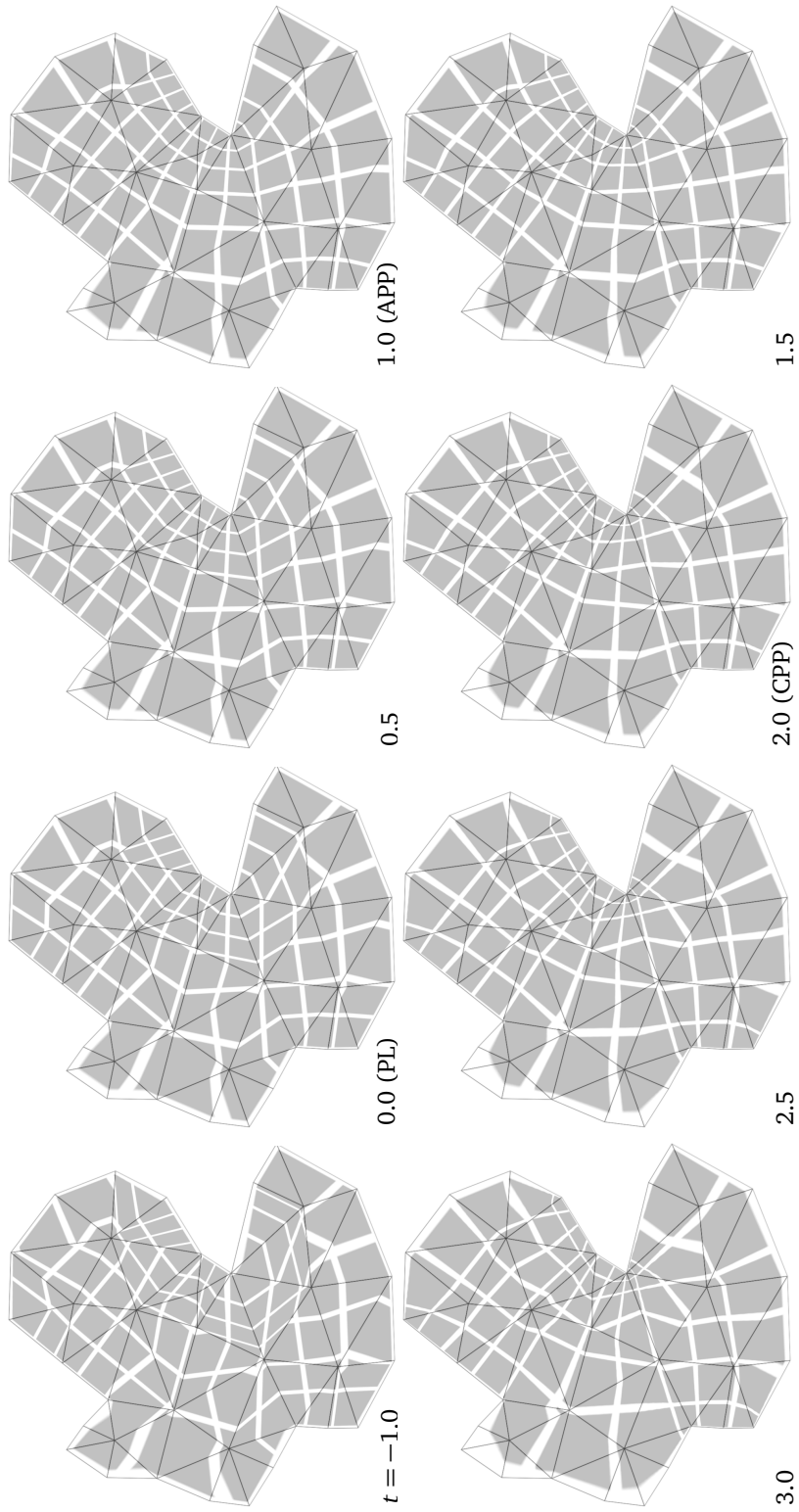


Figure 9: One-parameter family of piecewise projective interpolations.

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